A generalised Joyce construction for a family of nonlinear partial differential equations

S. K. Donaldson

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The construction of Dominic Joyce referred to in the title is for self-dual Riemannian metrics with two Killing fields [5]. In this note we show that this construction (in the form described by Calderbank and Pedersen in [2]) can be extended to a family of nonlinear fourth order PDE in two dimensions, essentially reducing them to linear equations. One equation in this family is the *affine maximal equation*, and we will see that the construction in this case is amost the same as one due to Chern and Terng.

1 The main result

We consider convex functions $u(x_i)$ defined on a domain in $\mathbf{R}^n$ and write $J = \det(u_{ij})$, where (u_{ij}) is the Hessian $\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$. Let ψ be any smooth, strictly convex function on the half-line $(0, \infty)$ and consider the functional

$$\mathcal{F} = \int \psi(J)\ dx_1 \dots dx_n. \tag{1}$$

The corresponding Euler-Lagrange equations $\delta\mathcal{F} = 0$ are

$$\sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left(J\psi'(J) u^{ij} \right) = 0, \tag{2}$$

where (u^{ij}) is the matrix inverse of (u_{ij}). This a nonlinear fourth order PDE for the function u. In the case when $\phi(J) = -J^\alpha$ for some $\alpha \in (0,1)$ these equations have been studied by Trudinger and Wang [6]. The case covered by Joyce's original construction is when $n = 2$ and $\psi(J) = -\log J$, as we will discuss further in Section 3. (Of course we only consider the functional $\mathcal{F}$ as a motivation for writing down the partial differential equations (2), and the actual convergence of the integral (1) is irrelevant.) Let us say that a point (x_1, x_2) is an "ordinary point" if the derivative ∇J does not vanish there.

To state our result, let f be a solution of the equation

$$f'(t) = t^{1/2}\psi''(t).$$

Thus $f' > 0$ and f is a $1-1$ map from $(0,\infty)$ to some finite or infinite interval I. Let p be the inverse map raised to the power $-1/2$, so $J^{-1/2} = p(r)$ when $r = f(J)$. Now consider the linear, second order PDE for a function $\xi(H,r)$ defined on some domain in $\mathbf{R} \times I$;

$$\frac{\partial^2 \xi}{\partial H^2} + \frac{1}{p(r)} \frac{\partial}{\partial r}\left(p(r)\frac{\partial \xi}{\partial r}\right) = 0. \tag{3}$$

Theorem 1 *Suppose ξ_1, ξ_2 are two solutions of equation (3) and* $\det\left(\frac{\partial(\xi_1,\partial\xi_2)}{\partial(H,r)}\right)$ *is positive at some point (H_0,r_0). Define three 1-forms $\epsilon_1,\epsilon_2,\epsilon$ by*

$$\epsilon_1 = p(r)(\frac{\partial \xi_2}{\partial r}dH - \frac{\partial \xi_2}{\partial H}dr) \quad \epsilon_2 = p(r)(-\frac{\partial \xi_1}{\partial r}dH + \frac{\partial \xi_1}{\partial H}dr)$$

$$\epsilon = \xi_1\epsilon_1 + \xi_2\epsilon_2.$$

Then $\epsilon,\epsilon_1,\epsilon_2$ are closed 1-forms and $\epsilon_1 \wedge \epsilon_2$ is non-vanishing near (H_0,r_0). Thus we can find functions u, x_1, x_2 with $du = \epsilon, dx_1 = \epsilon_1, dx_2 = \epsilon_2$ and x_1, x_2 give local co-ordinates around (H_0, r_0). If we regard u as a function of (x_1,x_2) it is a solution of the equation (2), on a suitable domain. All points in this domain are ordinary points. Conversely, any solution of (2) with $n = 2$, in the neighbourhood of an ordinary point, is obtained in this way, with solutions ξ_1,ξ_2 of (3) which are unique up to translation in the H-variable and the addition of constants

In sum, the local study of the nonlinear equation (2), in two dimensions, is essentially equivalent to that of the linear equation (3).

2 The proof

This is entirely elementary. From now on we always suppose the dimension n is 2, and we work locally so we will not specify the precise domains of definition of the various functions. Given a convex function u we consider the Riemannian metric $g = \sum u_{ij} dx_i dx_j$, and in particular the *conformal structure* which this defines. Write ξ_i for the Legendre transform co-ordinates $\xi_i = \frac{\partial u}{\partial x_i}$. Suppose now that we have some other local co-ordinates λ_1, λ_2, inducing the same orientation as x_1, x_2 (i.e. $\det\left(\frac{\partial x_i}{\partial \lambda_a}\right) > 0$). We can write the metric $g = \sum g_{ab} d\lambda_a d\lambda_b$ in these co-ordinates. Recall that the co-ordinates are called *isothermal* if the matrix (g_{ab}) is a multiple of the identity matrix at each point, or in other words

$$g = V(d\lambda_1^2 + d\lambda_2^2),$$

for a positive function $V(\lambda_1,\lambda_2)$. Let ϵ_{ij} denote the alternating tensor $\epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0$.

Observation 1 *The co-ordinates λ_1, λ_2 are isothermal if and only if the partial derivatives $\frac{\partial x_i}{\partial \lambda_a}, \frac{\partial \xi_j}{\partial \lambda_b}$ are related by the four equations*

$$\frac{\partial \xi_i}{\partial \lambda_a} = \epsilon_{ab}\epsilon_{ij}\sqrt{J}\frac{\partial x_j}{\partial \lambda_b}, \tag{4}$$

where $J = J(\underline{x}(\lambda_1, \lambda_2))$.

To see this we write

$$g = \sum_{ij} u_{ij} dx_i dx_j = \sum_j d\xi_j dx_j = \sum_{j,a,b} \frac{\partial \xi_j}{\partial \lambda_a}\frac{\partial x_j}{\partial \lambda_b} d\lambda_a d\lambda_b.$$

So the isothermal condition is

$$\sum_j \frac{\partial \xi_j}{\partial \lambda_a}\frac{\partial x_j}{\partial \lambda_b} = V\delta_{ab}.$$

In matrix notation, if $A = \left(\frac{\partial x_i}{\partial \lambda_a}\right), B = \left(\frac{\partial \xi_i}{\partial \lambda_a}\right)$, this is

$$A^T B = V 1,$$

or in other words

$$B^T = V A^{-1}. \tag{5}$$

Taking determinants we have $\det B \det A = V^2$. On the other hand the matrix (u_{ij}) is BA^{-1}, so $J = \det B \det A^{-1}$. Thus $V = \sqrt{J}\det A$ and (5) is

$$B^T = \sqrt{J}\det A \; A^{-1},$$

which is the same as (4), by the formula for the inverse of a 2×2 matrix.

Now consider any positive function $P(\lambda_1, \lambda_2)$ and the pair of second order linear PDE

$$\frac{\partial}{\partial \lambda_1}\left(P\frac{\partial \xi}{\partial \lambda_1}\right) + \frac{\partial}{\partial \lambda_2}\left(P\frac{\partial \xi}{\partial \lambda_2}\right) = 0, \tag{6}$$

$$\frac{\partial}{\partial \lambda_1}\left(P^{-1}\frac{\partial x}{\partial \lambda_1}\right) + \frac{\partial}{\partial \lambda_2}\left(P^{-1}\frac{\partial x}{\partial \lambda_2}\right) = 0. \tag{7}$$

Observation 2 *If $\xi(\lambda_1, \lambda_2)$ is a solution of (6) then there is a solution $x(\lambda_1, \lambda_2)$ of the first order system*

$$\frac{\partial x}{\partial \lambda_a} = P\epsilon_{ab}\frac{\partial \xi}{\partial \lambda_b}, \tag{8}$$

unique up to the addition of a constant, and $x(\lambda_1, \lambda_2)$ satisfies (7).

This is because the consistency condition for the first order system is

$$\frac{\partial}{\partial\lambda_2}\left(P\frac{\partial\xi}{\partial\lambda_2}\right)=\frac{\partial}{\partial\lambda_1}\left(-P\frac{\partial\xi}{\partial\lambda_1}\right),$$

which is the equation (6). In a different language, we are saying that the 1-form $\epsilon = P^{-1}\frac{\partial\xi}{\partial\lambda}{}_2 d\lambda_1 - P\frac{\partial\xi}{\partial\lambda_1}d\lambda_2$ is closed, so can be written as dx, for a function x. Note that, changing P to P^{-1}, there is a complete symmetry between x and ξ so we can also start with a solution of (7) and construct a solution of (6).

Observation 3 *Suppose ξ_1, ξ_2 are two solutions of (6), with* $\det\left(\frac{\partial\xi_i}{\partial\lambda_a}\right) > 0$. *Let x_2 be the solution of (7) corresponding to ξ_1 by (8), and x_1 be the solution corresponding to $-\xi_2$. Then* $\det\left(\frac{\partial x_i}{\partial\lambda_a}\right) > 0$*, so x_1, x_2 give local co-ordinates. Write*

$$\epsilon = \xi_1 dx_1 + \xi_2 dx_2.$$

Then ϵ is a closed 1-form and so $\epsilon = du$ for a function u. If we express u as a function of x_1, x_2 then $\frac{\partial u}{\partial x_i} = \xi_i$.

This is all straightforward. The conditions (8) imply that

$$\det\left(\frac{\partial x_i}{\partial\lambda_a}\right) = P^2 \det\left(\frac{\partial\xi_i}{\partial\lambda_a}\right) > 0.$$

We have $d\epsilon = d\xi_1 dx_1 + d\xi_2 dx_2$ and, writing $\xi_{i,a} = \frac{\partial\xi_i}{\partial\lambda_a}$,

$$d\epsilon = (\xi_{1,1}d\lambda_1+\xi_{1,2}d\lambda_2)(-\xi_{2,2}d\lambda_1+\xi_{2,1}d\lambda_2)+(\xi_{2,1}d\lambda_1+\xi_{2,2}d\lambda_2)(\xi_{1,2}d\lambda_1-\xi_{1,1}d\lambda_2) = 0.$$

Now return to our functions $\psi(J), f(J)$ and the Euler-Lagrange equation (2).

Observation 4 *A convex function u satisfies equation (2) if and only if $f(J)$ is harmonic with respect to the metric $g = \sum u_{ij} dx_i dx_j$.*

This is true in any dimension. The formula for the derivative of an inverse matrix is

$$\frac{\partial}{\partial x_k}u^{ij} = -\sum_{pq} u^{ip} u_{pqk} u_{qj},$$

whereas the formula for the derivative of the determinant is

$$\frac{\partial J}{\partial x_i} = J \sum_{pq} u^{pq} u_{pqi}.$$

These yield the identity

$$\sum_j \frac{\partial}{\partial x_j} u^{ij} = -\sum_{pqj} u^{ip} u_{jpq} u^{qj} = -\sum_p u^{ip} J^{-1} \frac{\partial J}{\partial x_p}. \qquad (9)$$

Our Euler-Lagrange equation (2) is

$$\sum_i \frac{\partial v_i}{\partial x_i} = 0, \tag{10}$$

where

$$v_i = \sum_j \frac{\partial}{\partial x_j}(J\psi'(J)u^{ij}).$$

So

$$v_i = \sum_j \frac{\partial(J\psi'(J))}{\partial x_j}u^{ij} + J\psi'(J)\frac{\partial u^{ij}}{\partial x_j}.$$

By the definition of the function f we have

$$\frac{\partial}{\partial x_j}(J\psi'(J)) = \sqrt{J}\frac{\partial f(J)}{\partial x_j} + \psi'(J)\frac{\partial J}{\partial x_j}.$$

Using (9) we obtain

$$v_i = \sqrt{J}\frac{\partial f(J)}{\partial x_j}u^{ij}.$$

Thus the equation (10) is the Laplace equation in the metric g:

$$\sum_i \frac{\partial}{\partial x_i}\left(\sqrt{J}u^{ij}\frac{\partial f(J)}{\partial x_j}\right) = 0.$$

With these four observations the main result, Theorem 1, is almost obvious. Suppose we start with an ordinary point of a solution u to (2). Then $r = f(J)$ is harmonic, by Observation 4, and we can suppose that its derivative does not vanish in the region considered. There is then a conjugate harmonic function H, which by definition is one such that the local co-ordinates (H, r) are isothermal. By Observations 1 and 2 the functions x_i, ξ_j satisfy the equations (6),(7) respectively with $\lambda_1 = H, \lambda_2 = r$ and $P = p(r)$. Since P does not depend on H the equation (6) can be written in the form (3). By Observations 2 and 3 we can recover the original function u from the two solutions ξ_1, ξ_2 of the linear PDE (or, equally well, the two solutions x_1, x_2). It is also clear that, conversely, starting with any two solutions ξ_1, ξ_2 to the linear equation we construct a solution to (2) by this method.

Note that the conjugate function H, in this situation, can be defined simply as the solution of the system

$$\frac{\partial H}{\partial x_i} = \epsilon_{ij}v_j, \tag{11}$$

so we can also thing of H as the Hamiltonian generating the area-preserving vector field v in the (x_1, x_2) plane. We leave this as an exercise for the reader.

3 Examples and discussion

1. Let $u^*(\xi_1, \xi_2)$ be the Legendre transform of $u(x_1, x_2)$. The Hessian $(\frac{\partial^2 u^*}{\partial \xi_i \partial \xi_j})$ at the point $\xi_i = \frac{\partial u}{\partial x_i}$ is the inverse of the Hessian of u at x_i. Thus we can write
$$\mathcal{F} = \int \psi(J_*^{-1}) J_* \; d\xi_1 d\xi_2,$$
where $J_* = \det(\frac{\partial^2 u^*}{\partial \xi_i \partial \xi_j})$. This implies that the Legendre transform takes a solution u of the equation (2) associated with ψ to a solution u^* of the equation associated to the function
$$\psi^*(t) = t\psi(t^{-1}).$$
In our construction this just means replacing the function $p(r)$ by $p(-r)^{-1}$ and interchanging the roles of the co-ordinates x_i, ξ_i.

2. If $\psi(t) = -\log t$ the equation (2) is $\sum \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = 0$. This is the equation defining a zero scalar curvature Kahler metric in "symplectic" co-ordinates, see [1], [4]. Explicitly we introduce two further co-ordinates θ_1, θ_2 and consider the Riemannian metric, in four dimensions,
$$\sum u_{ij} dx_i dx_j + \sum u^{ij} d\theta_i d\theta_j.$$
The well-known fact that, in two complex dimensions, such metrics are self-dual gives the link with Joyce's original formulation of his construction. Under the Legendre transform we get another description corresponding to the function $\psi^*(t) = t\log t$. This leads to the equations describing zero scalar curvature metrics in "complex" co-ordinates. When $\psi(t) = -\log t$ we get $p(r) = r$ and equation (3) is the familiar equation defining axi-symmetric harmonic functions on $\mathbf{R}^3$.

3. For any function $p(r)$ there is an obvious solution $\xi = H$ to (3). Thus we get a special family of solutions to (2) with $\xi_1 = H$ and ξ_2 some other solution of (3). There is another special family, which corresponds to this under the Legendre transform, when ξ_1 is a function of r only, so $x_2 = H$. These correspond to second order equations of Monge-Ampère type which give special solutions of (2). For example, in the zero scalar curvature case above we have the special solutions where $\xi_1 = \log r$. The function u satisfies the equation $\det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = e^{-\xi_1/2}$ and the corresponding four dimensional Riemannian metric is Ricci-flat.

4. For the Trudinger-Wang equations, when $\psi(t) = -t^\alpha$ with $0 < \alpha < 1/2$, we get $p(r) = r^{1/(1-2\alpha)}$ (up to factor which is irrelevant because it drops out of the equation (3)). If $\alpha = 1/2$ we get $p(r) = e^{r/2}$.

5. Consider the graph of a function $u(x_1, x_2)$ as a surface in $\mathbf{R}^3$. The Gauss curvature is

$$K = \frac{\det(u_{ij})}{(1 + |\nabla u|^2)^2} = \frac{J}{(1 + |\nabla u|^2)^2},$$

and the induced area form is

$$dA = (1 + |\nabla u|^2)^{1/2} dx_1 dx_2.$$

Thus

$$K^{1/4} dA = J^{1/4} dx_1 dx_2.$$

The left hand side is invariant under Euclidean transformations of $\mathbf{R}^3$ while the right hand side is invariant under unimodular affine transformations of $\mathbf{R}^2$. Since these two groups generate all unimodular affine transformations of $\mathbf{R}^3$ we see that this 2-form is an *affine* invariant of the surface. The graphs of the solutions of equation (2) when $\psi(t) = t^{1/4}$ are "affine maximal" surfaces in $\mathbf{R}^3$. According to Chern and Terng [3] these surfaces can be described locally as follows. Let F_1, F_2, F_3 be harmonic functions of variables λ_1, λ_2 and write $\underline{F} = \underline{F}(\lambda_1, \lambda_2)$ for the corresponding vector-valued function. Then the condition $\frac{\partial^2 \underline{F}}{\partial \lambda_1^2} + \frac{\partial^2 \underline{F}}{\partial \lambda_2^2} = 0$ implies the consistency of the first order system

$$\frac{\partial \underline{Z}}{\partial \lambda_1} = \underline{F} \times \frac{\partial \underline{F}}{\partial \lambda_2} \qquad \frac{\partial \underline{Z}}{\partial \lambda_2} = -\underline{F} \times \frac{\partial \underline{F}}{\partial \lambda_1} \tag{12}$$

so there is a solution $\underline{Z}$ which, under appropriate non-degeneracy conditions, parametrises a surface in $\mathbf{R}^3$. Chern and Terng show that these surfaces are precisely the affine maximal surfaces. We want to relate this description to ours. Notice first that our equations can be written in a similar form. Given a pair of solutions ξ_1, ξ_2 of (3) we define a vector valued function Ξ with components $\xi_1, \xi_2, 1$. Then if $\underline{Z} = (x_1, x_2, u)$ our description is the first order system

$$\frac{\partial \underline{Z}}{\partial H} = p(r)\ \Xi \times \frac{\partial \xi}{\partial r} \quad \frac{\partial \underline{Z}}{\partial r} = -p(r)\ \Xi \times \frac{\partial \xi}{\partial H}. \tag{13}$$

In the case when $\psi(t) = t^{1/4}$ we get $p(r) = r^2$ so the equation (3) is not the ordinary Laplace equation, in the variables (H, r). However given a function $F(H, r)$ we consider the first order system

$$p(r)\frac{\partial \xi}{\partial H} = r\frac{\partial F}{\partial H} \qquad p(r)\frac{\partial \xi}{\partial r} = r\frac{\partial F}{\partial r} - F. \tag{14}$$

If $p(r) = r^2$ one readily checks that this system is consistent, so there is a solution $\xi(H, r)$. The equation (3) for ξ is equivalent to the ordinary Laplace equation for F. Now take two harmonic functions F_1, F_2 and set $F_3(H, r) = r$. Then a few lines of calculation show that the system (12) is identical to the system (13), when in the latter we use the functions ξ_1, ξ_2 corresponding to F_1, F_2 by solving (14).

References

[1] M. Abreu *Kahler geometry of toric varieties and extremal metrics* Int. J. Math. **9** 641-651 1998

[2] Calderbank, D.M.J. and Pedersen, H. *Self-dual Einstein metrics with torus symmetry* Jour. Diff. Geom. **60** 485-521 2002

[3] Chern, S-S. and Terng, C. L. *An analogue of Bäcklund's Theorem in affine geometry* Rocky Mountain J. Math. **10** 105-124 1980

[4] Donaldson, S.K. *Kahler geometry on toric manifolds, and some other manifolds with large symmetry* Preprint arxiv DG 07052879

[5] Joyce,D.D. *Explicit construction of self-dual 4-manifolds* Duke Math. J. **77** 519-552 1995.

[6] Trudinger, N. S. and X-J. Wang, X-J. *Bernstein-Jörgens theorem for a fourth order partial differential equation* J. Partial Differential equations **15** 78-88 2002